

Exploratory Results on Fixed Points of Freezing Automata Networks with (α, β) -Type Functions

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Abstract. Freezing automata networks are discrete dynamical systems in which the local update rule permits only the transition from state 0 to state 1, making each activation irreversible. We study a parametric subfamily defined by two thresholds $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \leq \beta$: an inactive vertex activates if and only if the number of its active neighbours lies in the closed interval $[\alpha, \beta]$. Using extremal graph theory, we establish tight upper bounds on the number of fixed points for all connected graphs, identifying complete graphs K_n as extremal for $\alpha \leq 1$ and star graphs S_n for $\alpha = 2$. For four canonical families—paths P_n , cycles C_n , stars S_n , and complete graphs K_n —we derive exact closed-form and recursive formulas, uncovering connections to the Fibonacci, Lucas, and Tribonacci sequences. We complement these results with lower bounds via a correspondence with dominating sets. Finally, we introduce a five-level complexity classification formalising the memory cost of evaluating each formula.

Keywords: Freezing automata networks · Boolean networks · Extremal graph theory · Fixed points · Cellular automata

1 Introduction

Automata networks on graphs provide a versatile framework for modelling complex systems whose components interact locally and evolve simultaneously. A natural and widely studied constraint is *irreversibility*: once a vertex becomes active, it can never be deactivated. Dynamics with this property—known as *freezing dynamics*—were formally introduced by Goles, Ollinger, and Theyssier [4] and serve as natural models for epidemics with permanent immunity, biological cell differentiation, percolation, and certain neural-activation phenomena.

This paper studies the parametric family $\Gamma_{\alpha, \beta}$, in which an inactive vertex activates precisely when its number of active neighbours lies in $[\alpha, \beta]$. The computational complexity of this family was analysed in [5]. The special case $\alpha = 0$ (*Life Without Death*) is $\mathcal{P}$ -complete for the prediction problem [3]; this result concerns temporal reachability and does not directly bound the fixed-point counting problem studied here.

Our central objective is to characterise which graph topologies maximise the number of fixed points under $\Gamma_{\alpha,\beta}$, and to derive exact formulas for canonical families, via extremal graph theory [1].

Organisation. Section 2 introduces the model. Sections 3 and 4 derive tight upper and lower bounds. Section 5 provides exact formulas via transfer matrices and pattern-avoidance arguments. Section 6 introduces the complexity classification. Section 7 summarises findings and outlines open problems.

2 Model and Notation

Throughout, $G = (V, E)$ is a finite simple connected undirected graph with $|V| = n$. For a configuration $x \in \{0, 1\}^n$, the *active-neighbour count* of a vertex $v \in V$ is $|\{w \in V : vw \in E, x_w = 1\}|$. A vertex with $x_v = 0$ is *inactive*; with $x_v = 1$, *active*.

Definition 1 ((α, β) -threshold dynamics). *Let $\alpha, \beta \in \mathbb{N}_0$ with $\alpha \leq \beta$. The freezing threshold dynamics $\Gamma_{\alpha,\beta} : \{0, 1\}^n \rightarrow \{0, 1\}^n$ is defined by*

$$\Gamma_{\alpha,\beta}(x)_v = \begin{cases} 1 & \text{if } x_v = 1, \\ 1 & \text{if } x_v = 0 \text{ and } \alpha \leq |\{w \in V : vw \in E, x_w = 1\}| \leq \beta, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

A configuration $x \in \{0, 1\}^n$ is a fixed point of $\Gamma_{\alpha,\beta}$ on G if $\Gamma_{\alpha,\beta}(x) = x$. We denote the number of fixed points by $\varphi(\Gamma_{\alpha,\beta}, G)$.

Since state 1 is absorbing under rule (1), every orbit reaches a fixed point in at most n steps. We assume $n \geq \min\{\beta + 1, 4\}$ throughout; the finitely many smaller cases may be verified by direct enumeration.

The four graph families that arise naturally in our extremal analysis are shown in Figure 1: P_n (path), C_n (cycle), S_n (star with hub c), K_n (complete graph).

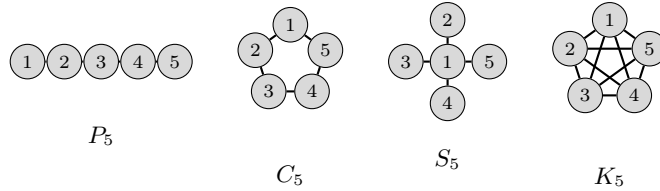


Fig. 1. The four graph families for $n = 5$. Each plays a distinct extremal role depending on (α, β) .

3 Extremal Upper Bounds

Theorem 1 (Upper bounds, $\alpha \leq 1$). *Let $\beta \geq 0$ and $n \geq \min\{\beta + 1, 4\}$. For every connected graph G on n vertices,*

$$\varphi(\Gamma_{0,\beta}, G) \leq 2^n - \sum_{i=0}^{\beta} \binom{n}{i}, \quad (2)$$

$$\varphi(\Gamma_{1,\beta}, G) \leq 2^n - \sum_{i=1}^{\beta} \binom{n}{i}. \quad (3)$$

Both bounds are tight, achieved by K_n and, more generally, by any $(\beta+1)$ -regular graph on n vertices.

Proof. Bound (2). If x has $k \leq \beta$ active vertices, every inactive v satisfies $|\{w \in V : vw \in E, x_w = 1\}| \leq k \leq \beta$, so $\Gamma_{0,\beta}(x)_v = 1 \neq x_v$ and x is not a fixed point. Subtracting the $\sum_{i=0}^{\beta} \binom{n}{i}$ such configurations gives (2).

Bound (3). If x has $1 \leq k \leq \beta$ active vertices, connectivity gives an edge $\{u, v\}$ with $x_u = 0, x_v = 1$, so $1 \leq |\{w \in V : uw \in E, x_w = 1\}| \leq k \leq \beta$ and $\Gamma_{1,\beta}(x)_u = 1 \neq x_u$; subtracting $\sum_{i=1}^{\beta} \binom{n}{i}$ gives (3).

Tightness. On K_n the active-neighbour count of any inactive vertex equals k , so a configuration with $k \notin [\alpha, \beta]$ is a fixed point and one with $k \in [\alpha, \beta]$ is not, yielding counts matching (2)–(3). The same holds for any $(\beta+1)$ -regular graph.

Theorem 2 (Upper bound, $\alpha = 2$). *Let $\beta \geq 2$ and $n_0(\beta) \leq 3(\beta+1)$. For every connected graph G on $n \geq n_0(\beta)$ vertices,*

$$\varphi(\Gamma_{2,\beta}, G) \leq 2^n - \sum_{i=2}^{\beta} \binom{n-1}{i}, \quad (4)$$

and this bound is tight, achieved by S_n . For $\alpha \geq 3$ and any $\beta \geq \alpha$, the trivial bound $\varphi(\Gamma_{\alpha,\beta}, G) \leq 2^n$ holds with equality on P_n .

Proof. Tightness on S_n . Each leaf ℓ_j has $|\{w \in V : \ell_j w \in E, x_w = 1\}| = x_c \leq 1 < \alpha = 2$, so no leaf activates. A configuration fails iff $x_c = 0$ and between 2 and β leaves are active, giving $\varphi(\Gamma_{2,\beta}, S_n) = 2^n - \sum_{i=2}^{\beta} \binom{n-1}{i}$.

Upper bound. For a vertex v of degree d , at least $\sum_{i=2}^{\beta} \binom{d}{i} \cdot 2^{n-d-1}$ configurations with $x_v = 0$ are non-fixed. The ratio $h_{\beta}(d) = \sum_{i=2}^{\beta} \binom{d}{i} / 2^d$ is unimodal and strictly decreasing for $d \geq n_0(\beta)$; if (4) were violated some degree would exceed $n-1$, a contradiction.

Case $\alpha \geq 3$. Every vertex of P_n has degree $\leq 2 < \alpha$, so $\varphi(\Gamma_{\alpha,\beta}, P_n) = 2^n$.

The values $n_0 = 7, 10, 10, 11$ for $\beta = 2, 3, 4, 5$ are confirmed by exhaustive computation.

4 Lower Bounds

Recall that $D \subseteq V$ is a *dominating set* of G if every vertex not in D has at least one neighbour in D . Let $\vartheta(G)$ denote the number of dominating sets of G .

Theorem 3 (Lower bounds, $\Gamma_{0,\beta}$). *Let $\beta \in \mathbb{N}_0$ and $n \geq \max\{4, \beta + 1\}$. For every connected graph G on n vertices,*

$$\varphi(\Gamma_{0,\beta}, G) \geq \begin{cases} \vartheta(G) & \beta = 0, \\ 2 & \beta = 1, \\ 1 & \beta \geq 2. \end{cases}$$

For $\beta = 0$, Wagner's result [6] gives $\varphi(\Gamma_{0,0}, G) \geq 5^{n/3}$ when $n \equiv 0 \pmod{3}$ (with analogous formulas for the other residues). None of these bounds can be improved.

Proof. *Case $\beta = 0$.* A configuration x is a fixed point of $\Gamma_{0,0}$ iff every inactive vertex v satisfies $|\{w \in V : vw \in E, x_w = 1\}| > 0$, i.e., the active set $\{v : x_v = 1\}$ is a dominating set. The lower bound follows from [6]. Tightness holds on graphs where every vertex of degree > 1 is adjacent to exactly two leaves.

Case $\beta = 1$. Since $n \geq 4$ and G is connected, there exists a vertex v with $\deg(v) \geq 2$. The all-ones configuration $x^{(1)}$ (where $x_w^{(1)} = 1$ for all w) and the configuration $x^{(v)}$ (where $x_v^{(v)} = 0$ and $x_w^{(v)} = 1$ for $w \neq v$) are both fixed points: in $x^{(v)}$, vertex v has $|\{w \in V : vw \in E, x_w^{(v)} = 1\}| = \deg(v) \geq 2 > \beta = 1$ active neighbours and is thus not activated. Equality holds on S_n : every fixed point must set all leaves to 1, after which $x_c \in \{0, 1\}$ freely.

Case $\beta \geq 2$. The all-ones configuration is always a fixed point.

5 Exact Formulas for Canonical Families

We use the following sequences: Fibonacci F_n ($F_1 = F_2 = 1$, OEIS A000045), Lucas L_n ($L_1 = 1, L_2 = 3$, OEIS A000204), path-Tribonacci T_n^P ($T_1^P = 1, T_2^P = 3, T_3^P = 5$, OEIS A000213), cycle-Tribonacci T_n ($T_1 = 1, T_2 = 3, T_3 = 7$, OEIS A001644). The discrepancy $T_3^P = 5$ vs. $T_3 = 7$ reflects the wrap-around constraint of C_n , which changes the initial conditions of the counting recurrences.

Theorem 4 (Complete graph). *For K_n with $n \geq \max\{4, \beta + 1\}$ and $0 \leq \alpha \leq \beta$, $\varphi(\Gamma_{\alpha,\beta}, K_n) = 2^n - \sum_{i=\alpha}^{\beta} \binom{n}{i}$.*

Proof. An inactive vertex v has $|\{w \in V : vw \in E, x_w = 1\}| = k$ when k vertices are active. The configuration x is a fixed point iff $k \notin [\alpha, \beta]$; otherwise every inactive vertex would be activated. Subtracting the $\sum_{i=\alpha}^{\beta} \binom{n}{i}$ excluded configurations from 2^n gives the result.

Example 1. $\varphi(\Gamma_{0,3}, K_4) = 16 - 15 = 1$; $\varphi(\Gamma_{2,3}, K_8) = 256 - \binom{8}{2} - \binom{8}{3} = 172$, in agreement with Table 1.

Theorem 5 (Star graph). For S_n with $n \geq \max\{4, \beta + 2\}$, hub c and leaves $\ell_1, \dots, \ell_{n-1}$:

$$\varphi(\Gamma_{\alpha, \beta}, S_n) = \begin{cases} 2^{n-1} + 1 & \alpha = 0, \beta = 0, \\ 2 & \alpha = 0, \beta \geq 1, \\ 2^{n-1} - \sum_{i=1}^{\beta} \binom{n-1}{i} + 1 & \alpha = 1, \beta \geq 1, \\ 2^n - \sum_{i=\alpha}^{\beta} \binom{n-1}{i} & \alpha \geq 2, \beta \geq 2. \end{cases}$$

Proof. Since every leaf ℓ_j has exactly one neighbour (c), its active-neighbour count equals $x_c \in \{0, 1\}$; leaves activate only when $x_c = 1$ and $1 \in [\alpha, \beta]$.

$\alpha = 0, \beta = 0$: With $x_c = 0$ the unique fixed point requires all leaves active (else a leaf with count 0 $\in [0, 0]$ activates). With $x_c = 1$ no inactive leaf activates (count 1 $\notin [0, 0]$), giving 2^{n-1} fixed points. Total: $2^{n-1} + 1$.

$\alpha = 0, \beta \geq 1$: Any inactive leaf has count $x_c \in [0, \beta]$ and activates; all leaves must be active. With all leaves active, c has $n - 1 \geq \beta + 1$ active neighbours and never activates regardless of x_c , giving exactly 2 fixed points.

$\alpha = 1, \beta \geq 1$: The all-active-leaves, $x_c = 1$ configuration is a fixed point (+1). With $x_c = 0$, c activates iff its active-leaf count is in $[1, \beta]$; valid configurations have 0 or $> \beta$ active leaves.

$\alpha \geq 2, \beta \geq 2$: No leaf activates (count $\leq 1 < \alpha$). The configuration fails iff $x_c = 0$ and the active-leaf count is in $[\alpha, \beta]$.

Theorem 6 (Cycles and paths). For $n \geq 4$, let $A_n = \sum_{i=0}^{\lfloor (n+2)/3 \rfloor} \binom{n+2-i}{2i}$ (OEIS A005251). Then $\varphi(\Gamma_{\alpha, \beta}, C_n)$ and $\varphi(\Gamma_{\alpha, \beta}, P_n)$ are given by

(α, β)	$\varphi(\Gamma_{\alpha, \beta}, C_n)$	$\varphi(\Gamma_{\alpha, \beta}, P_n)$
$\alpha = 0, \beta = 0$	T_n	T_n^P
$\alpha = 0, \beta = 1$	L_n	F_n
$\alpha = 0, \beta > 1$	1	1
$\alpha = 1, \beta = 1$	$L_n + 1$	$F_n + 1$
$\alpha = 1, \beta > 1$	2	2
$\alpha = 2, \beta \geq 2$	three-term recurrence, see (6)	
$\alpha \geq 3$	2^n	2^n

where for $\alpha = 2, \beta \geq 2$, both sequences satisfy

$$f(n) = 2f(n-1) - f(n-2) + f(n-3), \quad n \geq 6, \quad (6)$$

with initial conditions

$$\begin{aligned} \varphi(\Gamma_{2, \beta}, C_3) &= 5, & \varphi(\Gamma_{2, \beta}, C_4) &= 10, & \varphi(\Gamma_{2, \beta}, C_5) &= 17, \\ \varphi(\Gamma_{2, \beta}, P_3) &= 7, & \varphi(\Gamma_{2, \beta}, P_4) &= 12, & \varphi(\Gamma_{2, \beta}, P_5) &= 21. \end{aligned}$$

The closed form A_n (OEIS A005251) also gives $\varphi(\Gamma_{2, \beta}, P_n)$ directly.

The cases $\alpha \leq 1$ follow by pattern avoidance: a configuration on P_n is a fixed point of $\Gamma_{0,1}$ if and only if no inactive vertex has 0 or 1 active neighbours, equivalently, the binary string contains no isolated zero (pattern 101 is excluded, as are leading or trailing zeros adjacent to an active vertex with count $\leq \beta = 1$). The resulting count satisfies the Fibonacci recurrence. The case $\alpha = 2$ requires transfer matrices.

Proposition 1 (Transfer matrix, $\alpha = 2$). *Label P_n as $v_1, \dots, v_n$ and define: $E_n = \# \text{fixed points with } x_{v_n} = 1$; $S_n = \# \text{fixed points with } x_{v_n} = 0 \text{ and } x_{v_{n-1}} = 1$; $Z_n = \# \text{fixed points with } x_{v_n} = 0 \text{ and } (n = 1 \text{ or } x_{v_{n-1}} = 0)$; initial values $E_1 = 1, S_1 = 1, Z_1 = 0$. Then*

$$E_{n+1} = E_n + S_n, \quad S_{n+1} = S_n + Z_n, \quad Z_{n+1} = E_n, \quad (7)$$

and with $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,

$$\varphi(\Gamma_{2,\beta}, P_n) = [1 \ 1 \ 1] M^{n-1} [1 \ 1 \ 0]^\top, \quad \varphi(\Gamma_{2,\beta}, C_n) = \text{tr}(M^n). \quad (8)$$

The characteristic polynomial $x^3 - 2x^2 + x - 1 = 0$ of M yields via the Cayley–Hamilton theorem the three-term recurrence $f(n) = 2f(n-1) - f(n-2) + f(n-3)$, $n \geq 6$, satisfied directly by $\varphi(\Gamma_{2,\beta}, P_n)$ and $\varphi(\Gamma_{2,\beta}, C_n)$.

Proof. A configuration f on P_n is a fixed point unless some vertex v is inactive and both its neighbours are active (the unique forbidden pattern under $\alpha = 2$). Since v_{n+1} has degree 1, it can never have two active neighbours and is thus never forced to activate; we may freely set $x_{v_{n+1}} = 1$, giving $E_{n+1} = E_n + S_n$. For $x_{v_{n+1}} = 0$: if $x_{v_n} = 1$ (state S), the right neighbour of v_n is now 0, so v_n has at most one active neighbour and remains valid, giving $S_{n+1} = S_n + Z_n$; if $x_{v_n} = 0$ (state Z), setting $x_{v_{n+1}} = 0$ keeps v_n with zero active neighbours, and $Z_{n+1} = E_n$ records those extended paths ending $\dots 100$, where the new last two vertices are both inactive and the preceding vertex was active. The cycle formula $\text{tr}(M^n)$ enforces the wrap-around constraint by summing over all cyclic boundary conditions.

Example 2. Tables 1 and 2 illustrate a structural phase transition: $\varphi(\Gamma_{0,1}, C_{10}) = L_{10} = 123$ drops to $\varphi(\Gamma_{0,2}, C_{10}) = 1$ as β increases from 1 to 2. Under $\Gamma_{0,1}$ a zero vertex with two active neighbours ($> \beta = 1$) is *not* activated, sustaining shielded configurations; under $\Gamma_{0,2}$ it activates immediately, collapsing the fixed-point set to a single configuration and the count from $L_n \sim \phi^n$ to the constant 1.

Tables 1 and 2 provide numerical values for all four families, confirming the theoretical formulas.

Table 1. $\varphi(\Gamma_{\alpha,\beta}, G_n)$ for K_n and P_n .

n	K_n								P_n				
	(0,0)	(0,1)	(0,2)	(1,2)	(1,3)	(2,3)	(2,4)	(3,4)	(0,0)	(0,1)	(1,1)	(2,3)	(3,4)
4	15	11	5	6	2	6	5	11	9	3	4	12	16
6	63	57	42	43	23	29	14	29	31	8	9	37	64
8	255	247	219	220	164	172	102	130	105	21	22	114	256
10	1023	1013	968	969	849	859	649	694	355	55	56	351	1024

Table 2. $\varphi(\Gamma_{\alpha,\beta}, G_n)$ for S_n and C_n .

	S_n						C_n				
n	(0,0)	(0,1)	(1,2)	(1,3)	(2,3)	(3,4)	(0,0)	(0,1)	(1,1)	(2,3)	(3,4)
4	9	2	3	2	12	15	11	7	8	10	16
6	33	2	18	8	44	49	39	18	19	29	64
8	129	2	101	66	200	186	131	47	48	90	256
10	513	2	468	384	904	814	443	123	124	277	1024

6 Complexity Classification

The exact formulas of Section 5 vary in computational cost. We classify them by *memory requirement* (number of previously computed values needed) into five classes: **0** (constant, no memory), **I** (fixed-order linear recurrence), **II** (Class I plus a constant shift, e.g. $F_n + 1$ due to the all-zeros fixed point present when $\alpha \geq 1$), **III** (closed binomial expression, no memory), and **IV** (three-term recurrence or equivalent n -dependent sum). Table 3 summarises the partial classification obtained for the canonical graph families studied in this paper.

Table 3. Complexity class of $\varphi(\Gamma_{\alpha,\beta}, G_n)$.

Graph	(α, β) range	Class / formula
P_n, C_n, S_n	$\alpha \in \{0, 1\}, \beta > 1$	0 (const. 1 or 2)
P_n	$\alpha = 0, \beta = 1$	I (F_n , OEIS A000045)
C_n	$\alpha = 0, \beta = 0$	I (T_n , OEIS A001644)
P_n	$\alpha = 1, \beta = 1$	II ($F_n + 1$)
C_n	$\alpha = 1, \beta = 1$	II ($L_n + 1$)
K_n	all (α, β)	III ($2^n - \sum_{i=\alpha}^{\beta} \binom{n}{i}$)
S_n	$\alpha \geq 1, \beta \geq 1$	III ($2^n - \sum_{i=\alpha}^{\beta} \binom{n-1}{i}$)
P_n, C_n	$\alpha \geq 3, \beta \geq 3$	III (2^n)
C_n	$\alpha = 2, \beta \geq 2$	IV (rec. (6), init. 5, 10, 17)
P_n	$\alpha = 2, \beta \geq 2$	IV ($A_n = \sum_{i=0}^{\lfloor (n+2)/3 \rfloor} \binom{n+2-i}{2i}$)

7 Conclusion

We have studied the number of fixed points of (α, β) -freezing automata networks from an extremal graph-theoretic perspective, with three main contributions.

First, we established tight upper bounds for general connected graphs, identifying K_n as extremal for $\alpha \leq 1$ (Theorem 1) and S_n for $\alpha = 2$ (Theorem 2), complemented by lower bounds via dominating sets (Theorem 3). Second, exact formulas for K_n , S_n , C_n , and P_n reveal connections to Fibonacci, Lucas, and Tribonacci sequences and demonstrate a structural phase transition as β crosses 2. Third, a five-level complexity classification (Classes 0–IV) formalises the memory cost of fixed-point enumeration, showing that both graph topology and threshold parameters jointly determine the computational structure of the count.

Several directions remain open. First, a complete characterisation of extremal graphs beyond K_n and S_n is lacking: any $(\beta + 1)$ -regular graph also achieves the bound of Theorem 1, but the full extremal family has not been identified. Second, extending the lower bounds to general (α, β) via (α, β) -limited domination [2] and analysing majority-type threshold functions remain open problems.

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